SIMPLIFIED APPROACH TO THE PROBLEM OF THE OPTIMUM TRANSVERSAL CONTOUR (*) (**)

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The problem of finding the transversal contour of a conical body of given length and base area so as to minimize the total drag in hypersonic flow is considered under the assumptions that the pressure distribution is modified Newtonian and the surface-averaged skin-friction coefficient is constant. Both the case of a slender body and that of a nonslender body are investigated, and a simple proof of the properties of the extremal arc is supplied by solving first the local problem and then the integral problem. Specifically, the transversal contour locally minimizing the drag per unit base area is shown to be identical with the extremal contour. Depending on the length, the base area, and the skin-friction coefficient three solutions are possible: (1) a complete circle, (2) a combination of straight line segments tangent to a basic circle, and (3) a combination of circular arcs and straight line segments tangent to the circular arcs. For all of these solutions, the base area per unit perimeter and the aerodynamic drag per unit base area are constant along the extremal arc.

The problem of the optimum transversal contour of a body at hypersonic speeds has received considerable attention in recent years. After the pioneering work of Chernyi and Gonor on the minimization of the pressure drag [2], the minimization of the total drag has been considered by Miele and Saaris [3], Bellman [4], Reyn [5], and Miele and Hull [6].

In this paper, the minimization of the total drag of a hypersonic body of given length and base area is discussed once more, and a simple proof of the properties of the extremal arc is supplied by solving first the local problem and the the integral problem. Specifically, for both slender and nonslender bodies, it is shown that the transversal contour locally minimizing the drag per unit base area is identical with the extremal contour.

The following hypotheses are employed: (*a*) a plane of symmetry exists between the left-hand and right-hand sides of the body; (*b*) the base plane is perpendicular to the plane of symmetry; (*c*) the free-stream velocity is contained in the plane of symmetry and is perpendicular to the base plane; (*d*) the pressure coefficient is proportional to the cosine squared of the angle formed by the free-stream velocity and the normal to each surface element; (*e*) the base drag is neglected; (*f*) the skin-friction drag is proportional to the wetted area; and (*g*) the longitudinal contour is conical.

1. **Formulation of the problem.** We denote by D the drag, q the free-stream dynamic pressure, n a factor modifying the Newtonian pressure distribution, C_f the surface-averaged skin-friction coefficient (assumed constant), l the length, and S the base area, while θ and R are the polar coordinates of any point of the base. We introduce the constant

$$f = (C_f / 4n)^{1/3}$$

and define the dimensionless base radius, the drag parameter, and the area parameter as follows

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*) This research is a condensed version of the investigation described in paper [1]

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$$\rho = R / lf, \qquad y = D / 2\, nql^3f^4, \; K = S / l^2f^3$$

Next, in the light of hypotheses (*a*) to (*g*), we recognize that the problem of minimizing the drag for a given length and base area consists of extremizing the integral [3 to 6]

$$J = \int_0^\pi \left[\frac{\rho^6}{(\rho^2 + \rho^{\cdot 2} + mf^2\rho^4)} + 2\,(\rho^2 + \rho^{\cdot 2})^{1/2}\right] d\theta \tag{1.1}$$

where $\rho^\cdot = d\rho/d\theta$ and where $m = 0$ for a slender body and $m = 1$ for a nonslender body. The admissible functions $\rho(\theta)$ must satisfy the isoperimetric constraint

$$K = \int_0^\pi \rho^2\, d\theta \tag{1.2}$$

where K is a prescribed constant. The terminal radii $\rho(0)$ and $\rho(\pi)$ are free and must be found from the solution of the variational problem.

2. Local solution. Prior to solving the above integral problem, we study a related local problem. For any arbitrarily prescribed contour $\rho(\theta)$, we define the functions

$$J(\theta) = \int_0^\theta \left[\frac{\rho^6}{(\rho^2 + \rho^{\cdot 2} + mf^2\rho^4)} + 2\,(\rho^2 + \rho^{\cdot 2})^{1/2}\right] d\theta, \qquad K(\theta) = \int_0^\theta \rho^2\, d\theta \tag{2.1}$$

which are such that

$$J(0) = 0, \qquad K(0) = 0; \qquad J(\pi) = J, \qquad K(\pi) = K$$

Next, we introduce the derivatives $J^\cdot = dJ/d\theta$ and $K = dK/d\theta$ and observe that, because of Eqs. (2.1)

$$\frac{dJ}{d\theta} = \frac{\rho^6}{\rho^2 + \rho^{\cdot 2} + mf^2\rho^4} + 2\,(\rho^2 + \rho^{\cdot 2})^{1/2}, \qquad \frac{dK}{d\theta} = \rho^2$$

As a consequence, the aerodynamic drag per unit base area is proportional to

$$\frac{dJ}{dK} = \frac{\rho^4}{\rho^2 + \rho^{\cdot 2} + mf^2\rho^4} + 2\,(\rho^2 + \rho^{\cdot 2})^{1/2}$$

and can be rewritten as

$$\frac{dJ}{dK} = A(u), \qquad A(u) = \frac{u^2}{(1 + mf^2u^2)} + \frac{2}{u}, \qquad u = \frac{\rho^2}{(\rho^2 + \rho^{\cdot 2})^{1/2}} \tag{2.2}$$

The variable u is proportional to the enclosed area per unit perimeter and has the property that

$$u \leqslant \rho$$

Since the aerodynamic drag per unit enclosed area (2.2) depends on u only, we formulate the following problem: For each given radius ρ, find the parameter u which locally minimizes the aerodynamic drag per unit enclosed area (2.2). Of course, the admissible values of u are subordinated to the inequality constraint (2.3). The above problem belongs to the ordinary theory of maxima and minima and admits the solutions

$$u = \rho \quad (\rho \leqslant u_0) \qquad \text{or} \qquad u = u_0 \quad (\rho \geqslant u_0) \tag{2.4}$$

where u_0 is the value of u for which

$$dA / du = 0 \qquad \text{or} \qquad mf^2u_0^2 - u_0^{3/2} + 1 = 0 \tag{2.5}$$

Since the order of magnitude of the constant f is 10^{-1}, Eq. (2.5) can be solved by a linearization procedure in the neighborhood of $u_0 = 1$ to yield the approximate solution

$$u_0 \approx 1 + (2/3)\, mf^2 \tag{2.6}$$

For a slender body ($m = 0$), Eq. (2.6) reduces to $u_0 = 1$, an exact solution of Eq. (2.5) For a nonslender body ($m = 1$), the value of u_0 differs from that of a slender body by less than 1 %.

3. Integrated properties. Now, assume that a body is constructed in such a way that the local optimum conditions (2.4) are satisfied everywhere. The following questions arise. What is the geometry of the body? What is its aerodynamic drag?

Solutions of class 1. These solutions are governed by the relationship (2.4), which can be rewritten as

$$\rho^{\cdot} = 0 \tag{3.1}$$

Upon integrating and accounting for the constraint of given base area, we see that

$$\rho = \text{const} = \sqrt{K / \pi}$$

This is a complete circle characterized by the drag parameter

$$J = K^2 / (\pi + mKf^2) + 2\sqrt{\pi K}, \qquad K \leqslant \pi u_0^2$$

Solutions of class 2. These solutions are governed by the relationship (2.4), which can be rewritten as

$$\rho^2 + \rho^{\cdot 2} = \rho^4 / u_0^2 \quad \text{or} \quad (\rho / u_0)^2 + (\rho^{\cdot} / u_0)^2 = (\rho / u_0)^4 \tag{3.2}$$

Upon integrating, we obtain the relationship $\rho \cos(\theta - \text{const}) = u_0$; which is a straight line segment tangent to the basic circle of radius $\rho = u_0$. By combining several such segments, a closed body having the drag parameter

$$J = (2u_0 + \sqrt{u_0})\, K, \qquad K \geqslant \pi u_0^2 \tag{3.3}$$

can be generated.

Solutions of class 3. These solutions are obtained by combining subarcs governed by Eq. (3.1) and subarcs governed by Eq. (3.2). Because of the continuity requirement, the subarcs (3.1) are characterized by the radius $\rho = u_0$. Hence, these solutions consist of arcs of the basic circle $\rho = u_0$ and straight line segments tangent to these arcs. Once more, the drag parameter is represented by Eq. (3.3).

4. Variational solution. We now consider the problem of minimizing the integral (1.1) subject to the constraint (1.2). This isoperimetric problem with variable end points is equivalent to that of minimizing the integral

$$J^* = I - \lambda K = \int_0^{\pi} F(\rho, \rho^{\cdot}, \lambda)\, d\theta$$

in which the fundamental function F is defined as

$$F = \rho^2 [A(u) - \lambda] \tag{4.1}$$

Here, λ is an undetermined, constant Lagrange multiplier, $A(u)$ and $u(\rho, \rho^{\cdot})$ are defined by Eqs. (2.2).

Standard variational techniques show that the extremal arc must satisfy the following first integral of the Euler equation:

$$F - \rho^{\cdot} F_{\rho^{\cdot}} = C \tag{4.2}$$

and the natural boundary conditions

$$F_{\rho^{\cdot}} = 0 \tag{4.3}$$

at the initial and final points. In the event that discontinuities in the derivative $\rho^{\cdot}$ occur, the corner conditions

$$\Delta(F - \rho^{\cdot} F_{\rho^{\cdot}}) = 0, \qquad \Delta F_{\rho^{\cdot}} = 0 \tag{4.4}$$

must be satisfied at the junction of any pair of subarcs composing the extremal arc.

In the above relations, the derivative

$$F_{\rho^{\cdot}} = \rho^2 (dA / du)(\partial u / \partial \rho^{\cdot})$$

as can be seen by employing Eqs. (2.2) and (4.1).

While a straightforward mathematical solution of this variational problem cannot be obtained (see, for instance, [3], the absence of any preferential direction in the transversal plane has prompted this writer to think that the local solution represented by Eqs. (2.4) is identical with the variational solution. That this is the case can be seen from the following reasoning. Since $\partial u / \partial \rho^{\cdot}$ vanishes along the solution (2.4) and dA/du vanishes along the solution (2.4), the local solution has the property that

$$F_{\rho^{\cdot}} = 0$$

everywhere. Hence, it satisfies the natural boundary conditions (4.3) and the corner condition (4.4). After the first integral (4.2) and the corner condition (4.4) are rewritten in the form

$$F = C, \qquad \Delta F = 0 \tag{4.5}$$

the following question arises. For each given isoperimetric constant K, is it possible to find a pair of constants λ and C for which the first condition (4.5) becomes an identity? This is precisely the case if one chooses

$$\lambda = \left(\frac{\pi}{K}\right)^{1/2} + \frac{K(2\pi + mKf^2)}{(\pi + mKf^2)^2}, \quad C = \left(\frac{\pi}{K}\right)^{1/2} - \frac{K^2}{(\pi + mKf^2)^2}$$

for the solutions of class 1 and

$$\lambda = 2 / u_0 + \sqrt{u_0}, \quad C = 0$$

for the solutions of class 2 and class 3.

For the latter solutions, the corner condition (4.6) is satisfied providing every subarc composing the extremal arc is characterized by the integration constant $C = 0$.

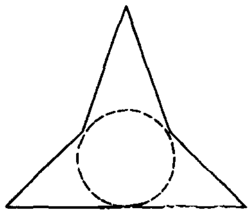

Fig. 1

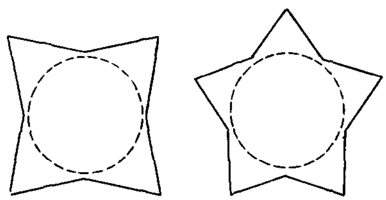

Fig. 2

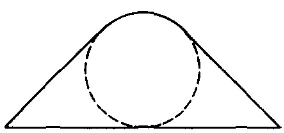

Fig. 3

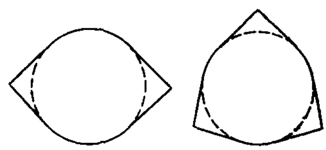

Fig. 4

In the previous sections, the minimization of the total drag of a conical body of given length and base area is discussed under the assumptions that the pressure distribution is modified Newtonian and the surface-averaged skin-friction coefficient is constant. Both the case of a slender body and that of a nonslender body are investigated, and a simple proof of the properties of the extremal arc is supplied by solving first a local problem and then the integral problem. Specifically, the transversal contour locally minimizing the drag per unit base area is shown to be identical with the extremal contour. Depending on the length, the base area, and the skin-friction coefficient (that is, depending on the area parameter K), three solutions are possible: (1) a complete circle; (2) a combination of straight line segments tangent to a basic circle (see Figs. 1 and 2); and (3) a combination of circular arcs (having the same radius and straight line segments tangent to the circular arcs (see Figs. 3 and 4). Each solution is characterized by the constancy of the base area per unit perimeter. An analogous remark holds for the aerodynamic drag per unit base area or unit perimeter. These invariant properties are due to the physics of the problem, that is, the absence of a preferential direction in the transversal plane.

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